

MAXIMAL INEQUALITY OF STOCHASTIC CONVOLUTION DRIVEN BY COMPENSATED POISSON RANDOM MEASURES IN BANACH SPACES

ZDZISŁAW BRZEŹNIAK*, ERIKA HAUSENBLAS* AND JIAHUI ZHU†

ABSTRACT. Assume that E is an M -type p Banach space with q -th, $q \geq p$, power of the norm is of C^2 -class. We consider the stochastic convolution

$$u(t) = \int_0^t \int_Z S(t-s) \xi(s, z) \tilde{N}(ds, dz),$$

where S is a C_0 -semigroup of contractions on E and \tilde{N} is a compensated Poisson random measure. We derive a maximal inequality for a càdlàg modification \tilde{u} of u

$$\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s)|_E^{q'} \leq C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q'}{p}},$$

for every $0 < q' < \infty$ and some constant $C > 0$. Stochastic convolution and M -type p Banach space and Poisson random measure

1. INTRODUCTION

The maximal inequality for stochastic convolutions of a contraction C_0 -semigroup and right continuous martingales in Hilbert spaces was studied by Ichikawa [8], see also Tubaro [13], for more details see [12]). A submartingale type inequality for the stochastic convolutions of a contraction C_0 -semigroup and square integrable martingales, also in Hilbert spaces, were obtained by Kotelenetz [10]. Kotelenetz also proved the existence of a càdlàg version of the stochastic convolution processes for square integrable càdlàg martingales. In the paper by Brzeźniak and Peszat [4], the authors established a maximal inequality in a certain class of Banach spaces for stochastic convolution processes driven by a Wiener process. It is of interest to know whether the maximal inequality holds also for pure jump type processes. Here we extend the results from [4] to the case where the stochastic convolution is driven by a compensated Poisson random measure. We work in the framework of stochastic integrals and convolutions driven by a compensated Poisson random measures recently introduced by the first two named authors in [3].

Let us now briefly present the content of the paper. In the first section, i.e. section 2 we set up notation and terminology and then summarize without proofs some of the standard facts on stochastic integrals with values in martingale type p , $p \in (1, 2]$, Banach spaces, driven by compensated Poisson random measures. In the following section 3, we proceed with the study of stochastic convolution process $(u(t))_{0 \leq t \leq T}$ driven by a compensated Poisson random measure \tilde{N} which is defined by the following formula

$$(1.1) \quad u(t) = \int_0^t \int_Z S(t-s) \xi(s, z) \tilde{N}(ds, dz), \quad t \in [0, T],$$

Date: May 11, 2010.

where $S(t)$, $t \geq 0$ is a contraction C_0 -semigroup on a martingale type p , $p \in (1, 2]$, Banach space E . In particular, we show that there exists a predictable version of this stochastic convolution process u . Under some suitable assumptions we show that the process u is a unique strong solution to the following stochastic evolution equation

$$(1.2) \quad \begin{aligned} du(t) &= Au(t)dt + \int_Z \xi(t, z) \tilde{N}(dt, dz), \quad t \in [0, T], \\ u(0) &= 0, \end{aligned}$$

where A is the infinitesimal generator of the contraction C_0 - semigroup $S(t)$, $t \geq 0$. In the last section 4 we present our main results. In particular, the maximal inequalities are stated and proved when the q -th power, for some q , of some equivalent norm on E is of C^2 class. We first show these inequalities for the exponent $q' \geq q$. Then we adapt some ideas from the paper of Ichikawa [8], see the proof of Theorem 1, and extend the maximal inequalities to the case of any q' in $(0, \infty)$. Thus, roughly speaking, we show that the process u has an E -valued càdlàg modification \tilde{u} which satisfies the following maximal inequality, see Theorems 4.4 and 4.5,

$$(1.3) \quad \mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s)|_E^{q'} \leq C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q'}{p}}, \quad t \in [0, T].$$

In the last part of section 4 we formulate and prove a different version of the maximal inequality.

2. STOCHASTIC INTEGRAL

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual hypothesis. Let (S, \mathcal{S}) be a measurable space. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Let $\mathbb{M}_{\bar{\mathbb{N}}}(S)$ denote the space of all $\bar{\mathbb{N}}$ -valued measures on (S, \mathcal{S}) . Let $\mathcal{B}(\mathbb{M}_{\bar{\mathbb{N}}}(S))$ be the smallest σ -field on $\mathbb{M}_{\bar{\mathbb{N}}}(S)$ with respect to which all the mapping $i_B : \mathbb{M}_{\bar{\mathbb{N}}}(S) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$, $B \in \mathcal{S}$ are measurable.

Definition 2.1. A Poisson random measure on (S, \mathcal{S}) over $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a map N such that the family $\{N(B) : B \in \mathcal{S}\}$ of random variables defined by $N(B) := i_B \circ N : \Omega \rightarrow \bar{\mathbb{N}}$ satisfies the following conditions

- (1) for any $B \in \mathcal{S}$, $N(B)$ is a random variable with Poisson distribution, i.e.

$$\mathbb{P}(N(B) = n) = e^{-\eta(B)} \frac{\eta(B)^n}{n!}, \quad n = 0, 1, 2, \dots,$$

with $\eta(B) = \mathbb{E}(N(B))$.

- (2) (independently scattered property) for any pairwise disjoint sets $B_1, \dots, B_n \in \mathcal{S}$, the random variables

$$N(B_1), \dots, N(B_n)$$

are independent.

Let (Z, \mathcal{Z}) be a measurable space. A point function on (Z, \mathcal{Z}) is a mapping $\alpha : \mathcal{D}(\alpha) \subset (0, \infty) \rightarrow Z$, where the domain $\mathcal{D}(\alpha)$ is a countable subset of $(0, \infty)$. Let Π_α be the set of all point functions on Z . Let \mathcal{Q} be the σ -field on Π_α generated by all mappings $\alpha \mapsto \#\{s \in (0, t] \cap \mathcal{D}(\alpha) : \alpha(s) \in A\}$, $A \in \mathcal{Z}$, $t > 0$.

Definition 2.2. We call a function $\pi : \Omega \rightarrow \Pi_\alpha$ a point process on Z if it is \mathcal{F}/\mathcal{Q} -measurable. A point process π is said to be stationary if for every $t > 0$, π and $\theta_t\pi$ have the same probability law, where $(\theta_t\pi)(s) = \pi(s+t)$, $\mathcal{D}(\theta_t\pi) = \{s \in (0, \infty) : s+t \in \mathcal{D}(\pi)\}$. For each point process π , we define a counting measure N_π by

$$N_\pi(t, A) := \sharp\{s \in (0, t] \cap \mathcal{D}(\pi) : \pi(s) \in A\}, \quad A \in \mathcal{Z}, \quad t \geq 0.$$

A point process π is called a Poisson point process if the counting measure N_π is a Poisson random measure. Moreover, a Poisson point process is σ -finite if there exists a sequence $\{D_n\}_{n \in \mathbb{N}} \subset \mathcal{Z}$ of increasing sets such that $\cup_n D_n = Z$ and $\mathbb{E}N_\pi(t, D_n) < \infty$ for all $0 < t \leq T$ and $n \in \mathbb{N}$. A Poisson point process is stationary if and only if there exists a nonnegative σ -finite measure on (Z, \mathcal{Z}) such that

$$\mathbb{E}N_\pi(t, A) = t\nu(A), \quad t \geq 0, \quad A \in \mathcal{Z}.$$

From now on, we suppose that π is a σ -finite stationary Poisson point process. For simplicity of notation, we write N instead of N_π . We employ the notation $\tilde{N}(t, A) = N(t, A) - t\nu(A)$, $t \geq 0$, $A \in \mathcal{Z}$ to denote the compensated Poisson random measure associated with the Poisson point process π . Let E be a real separable Banach space of martingale type p , $1 < p \leq 2$. That is there is a constant $K_p(E) > 0$ such that for all E -valued discrete martingale $\{M_n\}_{n=0}^N$ the following inequality holds

$$\sup_n \mathbb{E}|M_n|^p \leq K_p(E) \sum_{n=0}^N \mathbb{E}|M_n - M_{n-1}|^p$$

where we set $M_{-1} = 0$ as usual. Note that all L^q spaces, $q \geq p > 1$ are of martingale type p .

Definition 2.3. Let us fix $0 < T < \infty$. Let \mathcal{P} denote the σ -field on $[0, T] \times \Omega$ generated by all left-continuous and \mathcal{F}_t -adapted processes.

Let $\hat{\mathcal{P}}$ denote the σ -field on $[0, T] \times \Omega \times Z$ generated all functions $g : [0, T] \times \Omega \times Z \rightarrow E$ satisfying the following properties

- (1) for every $0 \leq t \leq T$, the mapping $(\omega, z) \mapsto g(t, \omega, z)$ is $\mathcal{Z} \otimes \mathcal{F}_t/\mathcal{B}(E)$ -measurable,
- (2) for every (ω, z) , the path $t \mapsto g(t, \omega, z)$ is left-continuous.

We say that an E -valued process $g = (g(t))_{0 \leq t \leq T}$ is predictable if the mapping $[0, T] \times \Omega \ni (t, \omega) \mapsto g(t, \omega) \in E$ is $\mathcal{P}/\mathcal{B}(E)$ -measurable.

We say that a function $f : [0, T] \times \Omega \times Z \rightarrow E$ is \mathbb{F} -predictable if the mapping is $\hat{\mathcal{P}}/\mathcal{B}(E)$ -measurable.

Proposition 2.4. $\hat{\mathcal{P}} = \mathcal{P} \otimes \mathcal{Z}$. Furthermore they are both equal to the σ -field generated by a family $\hat{\mathcal{R}}$ defined by

$$\hat{\mathcal{R}} = \{\{0\} \times F \times B, F \in \mathcal{F}_0, B \in \mathcal{Z}\} \cup \{(s, t] \times F \times B, F \in \mathcal{F}_s, B \in \mathcal{Z}, 0 \leq s < t \leq T\}.$$

Moreover, the family $\hat{\mathcal{R}}$ is a semiring.

Let $\mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$ denote the linear space of all \mathbb{F} -predictable functions $f : [0, T] \times \Omega \times Z \rightarrow E$ such that

$$\int_0^T \int_Z \mathbb{E} |f(t, z)|_E^p \nu(dz) dt < \infty.$$

In this section, we shall define for all functions f in the class $\mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$ the integral

$$\int_0^T \int_Z f(t, z) \tilde{N}(dt, dz),$$

which we shall call the stochastic integral with respect to a compensated Poisson random measure.

Definition 2.5. A function $f : [0, T] \times \Omega \times Z \rightarrow E$ is a step function if there is a finite sequence of numbers $0 = t_0 < t_1 < \dots < t_n = T$ and disjoint sets A_{j-1}^k , $j = 1, \dots, n$, $k = 1, \dots, m$, in \mathcal{Z} with $\nu(A_{j-1}^k) < \infty$ such that

$$(2.1) \quad f(t, \omega, z) = \sum_{j=1}^n \sum_{k=1}^m \xi_{j-1}^k(\omega) 1_{(t_{j-1}, t_j]}(t) 1_{A_{j-1}^k}(z),$$

where ξ_{j-1}^k is an E -valued p -integrable $\mathcal{F}_{t_{j-1}}$ -measurable random variable, $j = 1, \dots, n$ and $k = 1, \dots, m$. The class of all such step functions belonging to $\mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$ will be denoted by $\mathcal{M}_{step}^p([0, T] \times \Omega \times \mathcal{Z}; E)$.

Notice that a function of the form $1_{\{0\}}(t) \xi(t, \omega)$ with $\xi \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}} \otimes \mathcal{Z}, \lambda \times \mathbb{P} \times \nu; E)$ is equivalent to the identically zero process with respect to the measure $\lambda \times \mathbb{P} \times \nu$, so it has zero stochastic integral. Therefore, the inclusion or exclusion of the origin in the definition of step function is irrelevant.

Definition 2.6. The stochastic integral of a step function f in $\mathcal{M}_{step}^p([0, T] \times \Omega \times \mathcal{Z}; E)$ of the form (2.1) with respect to \tilde{N} is defined by, for $0 < t \leq T$,

$$I_t(f) := \sum_{j=1}^n \sum_{k=1}^m \xi_{j-1}^k(\omega) \tilde{N}((t_{j-1} \wedge t, t_j \wedge t] \times A_{j-1}^k).$$

Note that, for every $f \in \mathcal{M}_{step}^p([0, T] \times \Omega \times \mathcal{Z}; E)$, $I_t(f)$ does not depend on the representation (2.1) of the step function f and the process $I_t(f)$, $0 \leq t \leq T$ is a càdlàg martingale with mean 0. Moreover, $I_t(f)$ is linear with respect to f and satisfies the following inequality

$$(2.2) \quad \mathbb{E} |I_t(f)|_E^p \leq C \mathbb{E} \int_0^t \int_Z |f(s, z)|_E^p \nu(dz) ds,$$

where C , which is independent of the function f , is the same constant as the one in the martingale type p property of the space E . Let us now extend the definition of stochastic integral to all functions f in $\mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$. Take $f \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$. Then we can show that there exists a sequence $f^n \in \mathcal{M}_{step}^p([0, T] \times \Omega \times \mathcal{Z}; E)$ such that

$$\mathbb{E} \int_0^T \int_S \|f(t, \omega, z) - f^n(t, \omega, z)\|_E^p \nu(dz) dt \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

It follows from (2.2) that

$$\mathbb{E} |I_T(f^n) - I_T(f^m)|_E^p \leq C \mathbb{E} \int_0^t \int_Z |f^n(s, z) - f^m(s, z)|_E^p \nu(dz) ds \rightarrow 0,$$

as $n, m \rightarrow \infty$. In other words, $\{I_T(f^n)\}_{n=1}^\infty$ is a Cauchy sequence in $L^p(\Omega, E, \mathcal{F}_T)$. Thus the sequence $\{I_T(f^n)\}_{n=1}^\infty$ of random variables will converge in $L^p(\Omega, E, \mathcal{F}_T)$ to some particular random variable which we shall denote by $I_T(f)$. Moreover, such random variable is uniquely determined up to a set of measure zero in the variable ω . That is, it does not depend on the choice of the approximating step functions. We usually call $I_T(f)$ the stochastic integral of f with respect to a compensated Poisson random measure \tilde{N} . For $0 \leq a \leq b \leq T$, $B \in \mathcal{Z}$ and $f \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$, since $1_{(a, b]} 1_B f$ is also in $\mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$, so we can define the stochastic integral from a to b of the function $f \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$ by

$$I_{a, b}^B(f) = \int_a^b \int_B f(t, z) \tilde{N}(dt, dz) = I_T(1_{(a, b]} 1_B f).$$

For simplicity, we denote

$$I_t(f) = \int_0^t \int_Z f(t, z) \tilde{N}(dt, dz) = I_T(1_{(0, t]} f).$$

The following result was first proven in the case $p = 2$ in an important work [14] by Rüdiger.

Theorem 2.7. ([3]) *Let $f \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$. Then $I_t(f)$, $0 \leq t \leq T$ is a càdlàg p -integrable martingale with mean 0. More precisely, $I_t(f)$ has a modification which has \mathbb{P} -a.s. càdlàg trajectories. Moreover, it satisfies the following*

$$(2.3) \quad \mathbb{E} |I_t(f)|_E^p = \mathbb{E} \left| \int_0^t \int_Z f(s, z) \tilde{N}(ds, dz) \right|_E^p \leq C \mathbb{E} \int_0^t \int_Z |f(s, z)|_E^p \nu(dz) ds.$$

From now on, while considering the stochastic process $\int_0^t \int_Z f(s, z) \tilde{N}(ds, dz)$, $0 \leq t \leq T$, $f \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$, it will be assumed that the process $\int_0^t \int_Z f(s, z) \tilde{N}(ds, dz)$, $0 \leq t \leq T$, has \mathbb{P} -a.s. càdlàg trajectories.

3. STOCHASTIC CONVOLUTION

Let $(S(t))_{t \geq 0}$ be a contraction C_0 -semigroup on E . Suppose that A is the infinitesimal generator of the C_0 -semigroup $(S(t))_{t \geq 0}$. If $\{A_\lambda : \lambda > 0\}$ is the Yosida approximation of A , then for each λ , A_λ is a bounded operator in E and $|A_\lambda x - Ax|_E$ converges to 0 as $\lambda \rightarrow \infty$ for all $x \in E$, and uniformly convergence on bounded intervals. Let $R(\lambda, A) = (\lambda I - A)^{-1}$. By the use of Hille-Yosida Theorem (see [11]), it is easy to establish that $\lim_{\lambda \rightarrow \infty} \lambda R(\lambda, A)x = x$ and $\lambda R(\lambda, A)x \in \mathcal{D}(A)$, for all $x \in X$.

Let $\xi \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$. We are going to consider the following stochastic convolution process

$$(3.1) \quad u(t) = \int_0^t \int_Z S(t-s) \xi(s, z) \tilde{N}(ds, dz), \quad 0 \leq t \leq T,$$

where \tilde{N} is a compensated Poisson random measure of the point process $\pi = (\pi(t))_{t \geq 0}$.

We will first investigate the measurability of the process u .

Lemma 3.1. *The process $u(t)$, $0 \leq t \leq T$ given by (3.1) has a predictable version.*

Proof. Let $t \in [0, T]$ be fixed. We first show that a process X defined by $X(s) = 1_{(0,t]}(s)S(t-s)\xi(s, z)$, $0 \leq s \leq T$ is predictable. Define a function $F : [0, t] \times E \ni (s, x) \mapsto S(t-s)x \in E$. Since $S(t)$, $t \geq 0$ is a C_0 -semigroup, so for every $x \in E$, $F(\cdot, x)$ is continuous on $[0, t]$. Also, for every $s \geq 0$, $F(s, \cdot)$ is continuous. Indeed, let us fix $x_0 \in E$. Then for every $x \in E$, and $0 \leq t \leq T$,

$$|F(t, x) - F(t, x_0)|_E = |S(t-s)(x - x_0)|_E \leq |x - x_0|_E,$$

as $\|S(t)\|_{\mathcal{L}(E)} \leq 1$. This part shows that the function F is separably continuous. Since by assumption the process ξ is \mathbb{F} -predictable, one can see that the mapping

$$(s, \omega, z) \mapsto (s, \xi(s, \omega, z))$$

of $[0, T] \times \Omega \times Z$ into $[0, T] \times E$ is \mathbb{F} -predictable. Moreover, since the process $1_{(0,t]}$ is \mathbb{F} -predictable and we showed that the function F is separably continuous, so the composition mapping

$$(s, \omega, z) \mapsto (s, \xi(s, \omega, z)) \mapsto F(s, \xi(s, \omega, z)) \mapsto 1_{(0,t]}(s)F(s, \xi(s, \omega, z))$$

is \mathbb{F} -predictable as well. Therefore, process $X(s) = 1_{(0,t]}(s)F(s, \xi(s, z))$, $s \in [0, T]$ is \mathbb{F} -predictable. On the other hand, since $S(t)$, $t \geq 0$ is a C_0 -semigroup of contractions and ξ is in $\mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$, we have

$$\mathbb{E} \int_0^T |1_{(0,t]}S(t-s)\xi(s, z)|_E^p \nu(dz) ds \leq \mathbb{E} \int_0^T |\xi(s, z)|_E^p \nu(dz) ds < \infty.$$

Therefore, the process $1_{(0,t]}S(t-s)\xi(s, z)$ is of class $\mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$. Hence, when the number t is fixed, the integrals

$$\int_0^r \int_Z 1_{(0,t]}S(t-s)\xi(s, z)\tilde{N}(ds, dz), \quad r \in [0, T]$$

are well defined and by Theorem 2.7, this process is a martingale. In particular, for each $r \in [0, T]$, the integral $\int_0^r \int_Z 1_{(0,t]}S(t-s)\xi(s, z)\tilde{N}(ds, dz)$ is \mathcal{F}_r -measurable. Take $r = t$. This gives that $\int_0^t \int_Z 1_{(0,t]}S(t-s)\xi(s, z)\tilde{N}(ds, dz)$ is \mathcal{F}_t -measurable.

Now we show that the process u is continuous in p -mean. Observe that from the inequality $|a + b|^p \leq 2^p|a|^p + 2^p|b|^p$, inequality (2.3) and the contraction property of the semigroup $S(t)$, $t \geq 0$, we have, for $0 \leq r < t \leq T$,

$$\begin{aligned} \mathbb{E}|u(t) - u(r)|_E^p &= \mathbb{E} \left| \int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) - \int_0^r \int_Z S(r-s)\xi(s, z)\tilde{N}(ds, dz) \right|_E^p \\ &\leq 2^p \mathbb{E} \left| \int_r^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) \right|_E^p \\ &\quad + 2^p \mathbb{E} \left| \int_0^r \int_Z (S(t-s) - S(r-s))\xi(s, z)\tilde{N}(ds, dz) \right|_E^p \end{aligned}$$

$$\begin{aligned}
&\leq 2^p C_p \mathbb{E} \int_r^t \int_Z |S(t-s)\xi(s, z)|_E^p \nu(dz) ds \\
&\quad + 2^p C_p \mathbb{E} \int_0^r \int_Z |(S(t-s) - S(r-s))\xi(s, z)|_E^p \nu(dz) ds \\
&\leq 2^p C_p \mathbb{E} \int_r^t \int_Z |\xi(s, z)|_E^p \nu(dz) ds \\
&\quad + 2^p C_p \mathbb{E} \int_0^r \int_Z |(S(t-s) - S(r-s))\xi(s, z)|_E^p \nu(dz) ds \\
&= 2^p C_p \mathbb{E} \int_0^T \int_Z 1_{(r, t]}(s) |\xi(s, z)|_E^p \nu(dz) ds \\
&\quad + 2^p C_p \mathbb{E} \int_0^T \int_Z 1_{(0, r]}(s) |(S(t-s) - S(r-s))\xi(s, z)|_E^p \nu(dz) ds.
\end{aligned}$$

Here we note that $1_{(r, t]}(s) |\xi(s, z)|_E^p$ converges to 0 for all $(s, \omega, z) \in [0, T] \times \Omega \times Z$, as $t \downarrow r$ or $r \uparrow t$. So by the Lebesgue Dominated convergence theorem, the first term on the right hand side of above inequality converges to 0 as $t \downarrow r$ or $r \uparrow t$. For the second term, by the continuity of C_0 -semigroup $S(t)$, $t \geq 0$, the integrand $1_{(0, r]}(s) |(S(t-s) - S(r-s))\xi(s, z)|_E^p$ converges to 0 pointwise on $[0, T] \times \Omega \times Z$. Moreover we find that

$$|1_{(0, r]}(s) (S(t-s) - S(r-s))\xi(s, z)|_E \leq |\xi(s, z)|_E$$

So, again by the Lebesgue Dominated Convergence Theorem, the second term also converges to 0 as $t \downarrow r$ or $r \uparrow t$. Therefore, the process u is continuous in the p -mean. Since by Proposition 3.6 in [5], every adapted and stochastically continuous process on an interval $[0, T]$ has a predictable version on $[0, T]$, we conclude that the process $u(t)$, $0 \leq t \leq T$ has a predictable version. \square

Assume that A is the infinitesimal generator of a C_0 -semigroup $S(t)$, $t \geq 0$ of contractions on the Banach space E and that ξ is a function belonging to $\mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$.

Consider the problem (1.2) which for the convenience of the reader we write again below.

$$\begin{aligned}
(3.2) \quad &du(t) = Au(t)dt + \int_Z \xi(t, z) \tilde{N}(dt, dz) \\
&u(0) = 0,
\end{aligned}$$

Definition 3.2. Suppose that $\mathbb{E} \int_0^T \int_Z |\xi(s, z)|_E^p \nu(dz) dt < \infty$. A strong solution to Problem (1.2) is a $\mathcal{D}(A)$ -valued predictable stochastic process $(u(t))_{0 \leq t \leq T}$ such that

(1) $u(0) = 0$ a.s.

(2) For any $t \in [0, T]$ the equality

$$(3.3) \quad u(t) = \int_0^t Au(s) ds + \int_0^t \int_Z \xi(s, z) \tilde{N}(ds, dz)$$

holds \mathbb{P} -a.s..

Lemma 3.3. Let $\xi \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; \mathcal{D}(A))$. Then the process u defined by

$$(3.4) \quad u(t) = \int_0^t \int_Z S(t-s) \xi(s, z) \tilde{N}(ds, dz), \quad t \in [0, T],$$

is a unique strong solution of equation (1.2).

Proof. Let us fix $t \in [0, T]$. First we need to show that $u(t) \in \mathcal{D}(A)$. For this, Let $R(\lambda, A) = (\lambda I - A)^{-1}$, $\lambda > 0$, be the resolvent of A . Since $AR(\lambda, A) = \lambda R(\lambda, A) - I_E$, $AR(\lambda, A)$ is bounded. Hence, since $\xi \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; \mathcal{D}(A))$, we obtain

$$\begin{aligned} R(\lambda, A) \int_0^t \int_Z AS(t-s)\xi(s, z)\tilde{N}(ds, dz) &= \int_0^t \int_Z R(\lambda, A)AS(t-s)\xi(s, z)\tilde{N}(ds, dz) \\ &= \lambda R(\lambda, A) \int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) \\ &\quad - \int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} \int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) &= R(\lambda, A) \left[\lambda \int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) - \int_0^t \int_Z AS(t-s)\xi(s, z)\tilde{N}(ds, dz) \right]. \end{aligned}$$

Since $Rng(R(\lambda, A)) = \mathcal{D}(A)$, we infer that $\int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) \in \mathcal{D}(A)$. On the other hand, let us take $h \in (0, t)$ and observe that since $\frac{S(h)-I}{h}$ is bounded, we get the following equality

$$\begin{aligned} \frac{S(h)-I}{h} \int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) &= \int_0^t \int_Z \frac{S(h)-I}{h} S(t-s)\xi(s, z)\tilde{N}(ds, dz). \end{aligned}$$

So by applying the triangle inequality and inequality (2.3), we find that

$$\begin{aligned} \mathbb{E} \left| A \int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) - \int_0^t \int_Z AS(t-s)\xi(s, z)\tilde{N}(ds, dz) \right|^p &\leq 2^p \mathbb{E} \left| A \int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) - \frac{S(h)-I}{h} \int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) \right|^p \\ &\quad + 2^p \mathbb{E} \left| \int_0^t \int_Z AS(t-s)\xi(s, z)\tilde{N}(ds, dz) - \int_0^t \int_Z \frac{S(h)-I}{h} S(t-s)\xi(s, z)\tilde{N}(ds, dz) \right|^p \\ &\leq 2^p \mathbb{E} \left| \left(A - \frac{S(h)-I}{h} \right) \int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) \right|^p \\ &\quad + C_p \mathbb{E} \int_0^t \int_Z \left| AS(t-s)\xi(s, z) - \frac{1}{h} (S(h)-I) S(t-s)\xi(s, z) \right|_E^p \nu(dz) ds \\ (3.5) \quad &:= \text{I}(h) + \text{II}(h). \end{aligned}$$

For the integrand of $\text{I}(h)$, since $\xi(s, z) \in \mathcal{D}(A)$, we observe that $\frac{S(h)-I}{h} S(t-s)\xi(s, z) = \frac{1}{h} \int_0^h S(r)AS(t-s)\xi(s, z)dr$, so we have $\left| \frac{S(h)-I}{h} S(t-s)\xi(s, z) \right|_E^p \leq |A\xi(s, z)|_E^p$. Hence we infer that the integrand

$$\left| AS(t-s)\xi(s, z) - \frac{1}{h} (S(h)-I) S(t-s)\xi(s, z) \right|_E^p$$

of $\text{I}(h)$ is bounded by a function $C_1 |A\xi(s, z)|_E^p$ which is in $\mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$ by assumption. Since A is the infinitesimal generator of the C_0 -semigroup $S(t)$, $t \geq 0$, the integrand

$$\left| AS(t-s)\xi(s, z) - \frac{1}{h} (S(h)-I) S(t-s)\xi(s, z) \right|_E^p$$

converges to 0 pointwisely on $[0, t] \times \Omega \times Z$. Therefore, by the Lebesgue Dominated convergence theorem, the term $\text{II}(h)$ of above inequality (3.5) converges to 0 as $h \downarrow 0$.

Since we have already shown that $\int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) \in \mathcal{D}(A)$, it is easy to see that the term $\text{I}(h)$ of (3.5) converges to 0 as $h \downarrow 0$ as well. Hence by inequality (3.5) we conclude that

$$(3.6) \quad A \int_0^t \int_Z S(t-s)\xi(s, z)\tilde{N}(ds, dz) = \int_0^t \int_Z AS(t-s)\xi(s, z)\tilde{N}(ds, dz), \quad \mathbb{P}\text{-a.s.}$$

In order to verify equality (3.3), by the Fubini's theorem and equality (3.6) we find that

$$\begin{aligned} \int_0^t Au(s) ds &= \int_0^t \int_0^s \int_Z AS(s-r)\xi(r, z)\tilde{N}(dr, dz) ds \\ &= \int_0^t \int_Z \int_r^t AS(s-r)\xi(r, z) ds \tilde{N}(dr, dz) \\ &= \int_0^t \int_Z \int_r^t \frac{dS(s-r)\xi(r, z)}{ds} ds \tilde{N}(dr, dz) \\ &= \int_0^t \int_Z (S(t-r)\xi(r, z) - \xi(r, z)) \tilde{N}(dr, dz) \\ &= \int_0^t \int_Z S(t-r)\xi(r, z)\tilde{N}(dr, dz) - \int_0^t \int_Z \xi(r, z)\tilde{N}(dr, dz) \\ &= u(t) - \int_0^t \int_Z \xi(r, z)\tilde{N}(dr, dz), \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

which shows equality (3.3).

For the uniqueness, suppose that u^1 and u^2 are two strong solutions of Problem (1.2). Let $w = u^1 - u^2$. Then we infer that

$$w(t) = u^1(t) - u^2(t) = \int_0^t A(u^1(s) - u^2(s)) ds = A \int_0^t w(s) ds.$$

Put $v(t) = \int_0^t w(s) ds$. Then $v(t)$ is continuously differentiable on $[0, T]$ and $v(t) \in \mathcal{D}(A)$. Now applying Itô's formula to the function $f(s) = S(t-s)v(s)$ yields

$$\begin{aligned} \frac{df(s)}{ds} &= -AS(t-s)v(s) + S(t-s)\frac{dv(s)}{ds} \\ &= -AS(t-s)v(s) + S(t-s)w(s) = -AS(t-s)v(s) + S(t-s)Av(s) = 0. \end{aligned}$$

So we infer $v(t) = f(t) = f(0) = S(t)v(0) = 0$ a.s.. Therefore, $w(s) = 0$ a.s.. That is $u^1(t) = u^2(t)$ a.s. $t \in [0, T]$. \square

4. MAXIMAL INEQUALITIES FOR STOCHASTIC CONVOLUTION

Assumption 4.1. Suppose that E is a real separable Banach space of M -type p , $1 < p \leq 2$. In addition we assume that the Banach space E satisfies the following condition:

(Cond. 1) There exists an equivalent norm $|\cdot|_E$ on E and $q \in [p, \infty)$ such that the function $\phi : E \ni x \mapsto |x|_E^q \in \mathbb{R}$, is of class C^2 and there exists constant k_1, k_2 such that for every $x \in E$, $|\phi'(x)| \leq k_1|x|_E^{q-1}$ and $|\phi''(x)| \leq k_2|x|_E^{q-2}$.

Remark 4.1. It can be proved that if E satisfies condition **(Cond. 1)** for some q and $q_2 > q$, then E satisfies condition **(Cond. 1)** for q_2 .

Remark 4.2. Notice that the Sobolev space $H^{s,p}$ with $p \in [2, \infty)$ and $s \in \mathbb{R}$ satisfies above condition Cond. 1 and L^r -spaces with $r \geq q$ also satisfies condition Cond. 1.

Now we proceed with the study of the stochastic convolution

$$(4.1) \quad u(t) = \int_0^t \int_Z S(t-s) \xi(s, z) \tilde{N}(ds, dz), \quad t \in [0, T].$$

Before proving the main theorem, we first need the following Lemmas.

Lemma 4.2. For all $x \in D(A)$, $\phi'(x)(Ax) \leq 0$.

Proof. Take $0 \leq r < t < \infty$. We have

$$\begin{aligned} |S(t)x|_E^q - |S(r)x|_E^q &= |S(t-r)S(r)x|_E^q - |S(r)x|_E^q \\ &\leq |S(t-r)|_{\mathcal{L}(E)}^q |S(r)x|_E^q - |S(r)x|_E^q \\ &\leq |S(r)x|_E^q - |S(r)x|_E^q = 0, \quad \text{for all } x \in E. \end{aligned}$$

Thus the function $t \mapsto \phi(x)(S(t)x)$ is decreasing. Also, observe that for $x \in D(A)$,

$$\left. \frac{d\phi(S(t)x)}{dt} \right|_{t=0} = \phi'(S(0)x)(Ax) = \phi'(x)(Ax).$$

Hence $\phi'(x)(Ax) = \left. \frac{d\phi(S(t)x)}{dt} \right|_{t=0} \leq 0$ which shows the Lemma. \square

Lemma 4.3. The random variable $\sup_{0 \leq t \leq T} |u(t)|$ is measurable.

Proof. Since we have shown in Lemma 3.1 the stochastically continuity of the process u , applying Theorem 5.3 in [17], we can find a version \tilde{u} of u which is separable. That is there exists a countable subset T_0 which is everywhere dense in $[0, T]$ such that $\tilde{u}(t)$ belongs to the set of partial limits $\lim_{s \in T_0, s \rightarrow t} \tilde{u}(s)$ for all $t \in [0, T] \setminus T_0$. Hence

$$\sup_{t \in [0, T]} |\tilde{u}(t)| = \sup_{t \in [0, T]} \lim_{s_n \rightarrow t, s_n \in T_0} |\tilde{u}(s_n)| = \sup_{s_n \in T_0} |\tilde{u}(s_n)|,$$

where $\sup_{s_n \in T_0} |\tilde{u}(s_n)|$ is measurable. Therefore, the random variable $\sup_{t \in [0, T]} |\tilde{u}(t)|$ is also measurable. \square

Henceforth, when we study the stochastic convolution process u , we refer to the version of u such that it is predictable and its supremum over $[0, T]$ is measurable.

Theorem 4.4. Suppose E is an M -type p Banach space satisfying Assumption 4.1. Then there exists a càdlàg modification \tilde{u} of u and a constant C such that for every $0 < t \leq T$,

$$(4.2) \quad \mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s)|_E^{q'} \leq C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q'}{p}},$$

where $q' \geq q$ and q is the number from Assumption 4.1.

From now on, A denotes the infinitesimal generator of the C_0 -semigroup $(S(t))_{t \geq 0}$ of contractions.

Proof. Case I. First suppose that $\xi \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; \mathcal{D}(A))$. We will prove

$$(4.3) \quad \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^q \leq C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}},$$

We have shown in Lemma 3.3 that the process u is a unique strong solution to the following problem

$$(4.4) \quad \begin{aligned} du(t) &= Au(t)dt + \int_Z \xi(t, z) \tilde{N}(dt, dz), \quad t \in [0, T], \\ u(0) &= 0. \end{aligned}$$

Moreover, it can be written as

$$(4.5) \quad u(t) = \int_0^t Au(s) ds + \int_0^t \int_Z \xi(s, z) \tilde{N}(ds, dz), \quad t \in [0, T].$$

We shall note here that in view of the right continuity of the right hand side of (4.5), the càdlàg property of the function $u(t)$, $0 \leq t \leq T$ follows immediately. Notice that the function $\phi : E \ni x \mapsto |x|_E^q$ is of C^2 class by assumption. Thus, one may apply the Itô formula [7] to the process u and get for $t \in [0, T]$,

$$(4.6) \quad \begin{aligned} \phi(u(t)) &= \int_0^t \phi'(u(s))(Au(s)) ds + \int_0^t \int_Z \phi'(u(s-))(\xi(s, z)) \tilde{N}(ds, dz) \\ &\quad + \int_0^t \int_Z \left[\phi(u(s-) + \xi(s, z)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s, z)) \right] N(ds, dz) \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Since by Lemma $\phi'(x)(Ax) \leq 0$, for all $x \in D(A)$, we infer that for $t \in [0, T]$,

$$(4.7) \quad \begin{aligned} \phi(u(t)) &\leq \int_0^t \int_Z \phi'(u(s-))(\xi(s, z)) \tilde{N}(ds, dz) \\ &\quad + \int_0^t \int_Z \left[\phi(u(s-) + \xi(s, z)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s, z)) \right] N(ds, dz) \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Taking the supremum over the set $[0, t]$ and then the expectation to both sides of above inequality yields

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} \phi(u(s)) &\leq \mathbb{E} \sup_{0 \leq s \leq t} \int_0^s \int_Z \phi'(u(r))(\xi(r, z)) \tilde{N}(dr, dz) \\ &\quad + \mathbb{E} \sup_{0 \leq s \leq t} \int_0^s \int_Z \left[\phi(u(r-) + \xi(r, z)) - \phi(u(r-)) - \phi'(u(r-))(\xi(r, z)) \right] N(dr, dz) \\ &=: I_1(t) + I_2(t). \end{aligned}$$

Applying the Davis inequality and the Jensen inequality to I_1 we obtain for some constant C that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} I_1(s) &\leq C \mathbb{E} \left(\int_0^t \int_Z |\phi'(u(s-))(\xi(s, z))|^p N(ds, dz) \right)^{\frac{1}{p}} \\ &\leq k_1 C \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|^{q-1} \left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{1}{p}}. \end{aligned}$$

Firstly we are going to estimate the integral $I_2(t)$. Note that for every $s \in [0, t]$,

$$\begin{aligned} &\int_0^s \int_Z \left| \phi(u(r) + \xi(r, z)) - \phi(u(r)) - \phi'(u(r-))(\xi(r, z)) \right|_E N(dr, dz) \\ &= \sum_{r \in (0, s] \cap \mathcal{D}(\pi)} \left| \phi(u(r-) + \xi(r, \pi(r))) - \phi(u(r-)) - \phi'(u(r-))(\xi(r, \pi(r))) \right|_E, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

Let us recall that by the assumption the function ϕ is of C^2 class. Applying the mean value Theorem, see [9], to the function ϕ , for each $r \in [0, s]$ we can find $0 < \theta < 1$ such that

$$\left| \phi(u(r-) + \xi(r, \pi(r))) - \phi(u(r-)) \right|_E = |\xi(r, \pi(r))|_E \left| \phi'(u(r-) + \theta \xi(r, \pi(r))) \right|_{\mathcal{L}(E)}.$$

By the assumptions $|\phi'(x)| \leq k_1|x|_E^{q-1}$, $x \in E$ and the fact that $|x + \theta y|_E \leq \max\{|x|_E, |x + y|_E\}$ for all $x, y \in E$, we obtain

$$\begin{aligned} \left| \phi'(u(r-) + \theta \xi(r, \pi(r))) \right|_{\mathcal{L}(E)} &\leq k_1 |u(r-) + \theta \xi(r, \pi(r))|_E^{q-1} \\ &\leq k \max \{ |u(r-)|_E^{q-1}, |u(r-) + \xi(r, \pi(r))|_E^{q-1} \}. \end{aligned}$$

Observe that for all $0 \leq r \leq s \leq t$,

$$|u(r-)|_E^{q-1} \leq \sup_{0 \leq \rho \leq s} |u(\rho-)|_E^{q-1} \leq \sup_{0 \leq \rho \leq t} |u(\rho-)|_E^{q-1} = \sup_{0 \leq \rho \leq t} |u(\rho)|_E^{q-1}.$$

Moreover, since $u(r-) + \xi(r, \phi(r)) = u(r)$, we get

$$|u(r-) + \xi(r, \pi(r))|_E^{q-1} \leq \sup_{0 \leq r \leq s} |u(r)|_E^{q-1} \leq \sup_{0 \leq s \leq t} |u(s)|_E^{q-1}.$$

Therefore, we infer that for each $r \in [0, s]$,

$$\left| \phi(u(r-) + \xi(r, \pi(r))) - \phi(u(r-)) \right|_E \leq k_1 |\xi(r, \pi(r))|_E \sup_{0 \leq s \leq t} |u(s)|_E^{q-1}.$$

It follows that

$$\begin{aligned} &\left| \phi(u(r-) + \xi(r, \pi(r))) - \phi(u(r-)) - \phi'(u(r-))(\xi(r, \pi(r))) \right|_E \\ &\leq \left| \phi(u(r) + \xi(r, \pi(r))) - \phi(u(r)) \right|_E + \left| \phi'(u(r-))(\xi(r, \pi(r))) \right|_E \\ &\leq 2k_1 |\xi(r, \pi(r))|_E \sup_{0 \leq s \leq t} |u(s)|_E^{q-1}. \end{aligned}$$

On the other hand side, we can also find some $0 < \delta < 1$ such that

$$\begin{aligned} \left| \phi(u(r-) + \xi(r, \pi(r))) - \phi(u(r-)) - \phi'(u(r-))(\xi(r, \pi(r))) \right|_E &= \frac{1}{2} |\xi(r, \pi(r))|_E^2 |\phi''(u(r-) + \theta \xi(r, \pi(r)))| \\ &\leq \frac{k_2}{2} |\xi(r, \pi(r))|_E^2 |u(r-) + \theta \xi(r, \pi(r))|_E^{q-2}. \end{aligned}$$

A similar argument as above, we obtain

$$\left| \phi(u(r-) + \xi(r, \pi(r))) - \phi(u(r-)) - \phi'(u(r-))(\xi(r, \pi(r))) \right|_E \leq \frac{k_2}{2} |\xi(r, \pi(r))|_E^2 \sup_{0 \leq s \leq t} |u(s)|_E^{q-2}.$$

Summing up, we have

$$\begin{aligned} &\left| \phi(u(r-) + \xi(r, \pi(r))) - \phi(u(r-)) - \phi'(u(r-))(\xi(r, \pi(r))) \right|_E \\ &= \left| \phi(u(r-) + \xi(r, \pi(r))) - \phi(u(r-)) - \phi'(u(r-))(\xi(r, \pi(r))) \right|_E^{(2-p)+(p-1)} \\ &\leq \left(2k_1 |\xi(r, \pi(r))|_E \sup_{0 \leq s \leq t} |u(s)|_E^{q-1} \right)^{2-p} \left(\frac{k_2}{2} |\xi(r, \pi(r))|_E^2 \sup_{0 \leq s \leq t} |u(s)|_E^{q-2} \right)^{p-1} \\ &\leq K |\xi(r, \pi(r))|_E^p \sup_{0 \leq s \leq t} |u(s)|_E^{q-p}, \end{aligned}$$

where $K = (2k_1)^{2-p}(k_1/2)^{p-1}$.

Hence,

$$\begin{aligned} \sum_{r \in (0, t] \cap \mathcal{D}(\pi)} \left| \phi(u(r-) + \xi(r, \pi(r))) - \phi(u(r-)) - \phi'(u(r-))(\xi(r, \pi(r))) \right|_E \\ \leq K \sup_{0 \leq s \leq t} |u(s)|_E^{q-p} \sum_{r \in (0, t] \cap \mathcal{D}(\pi)} |\xi(r, \pi(r))|_E^p \\ = K \sup_{0 \leq s \leq t} |u(s)|_E^{q-p} \int_0^s \int_Z |\xi(r, z)|_E^p N(dr, dz), \end{aligned}$$

which also shows that the integral $\int_0^t \int_Z \left[\phi(u(s-) + \xi(s, z)) - \phi(u(s-)) - \phi'(u(s-))(\xi(s, z)) \right] N(ds, dz)$ is well defined since $\xi \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; \mathcal{D}(A))$. Therefore, we infer

$$\begin{aligned} \int_0^s \int_Z \left| \phi(u(r-) + \xi(r, z)) - \phi(u(r-)) - \phi'(u(r-))(\xi(r, z)) \right|_E N(dr, dz) \\ \leq K \sup_{0 \leq s \leq t} |u(s)|_E^{q-p} \int_0^s \int_Z |\xi(r, z)|_E^p N(dr, dz). \end{aligned}$$

Hence, we get the following estimate for $I_2(t)$

$$I_2(t) \leq K \sup_{0 \leq s \leq t} |u(s)|_E^{q-p} \int_0^s \int_Z |\xi(r, z)|_E^p N(dr, dz), \quad t \in [0, T],$$

where the constant K only depends on k_1 , k_2 , p and q . Now applying Hölder's and Young's inequalities to $I_1(t)$ yields

$$\begin{aligned} I_1(t) &\leq k_1 C \left[\left(\mathbb{E} \left(\sup_{0 \leq s \leq t} |u(s)|_E^{q-1} \right)^{\frac{q-1}{q}} \right)^{\frac{q-1}{q}} \left(\mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \right] \\ &\leq k_1 C \left[\left(\mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^q \right)^{\frac{q-1}{q}} \left(\mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \right] \\ &= k_1 C \left[\left(\mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^q \varepsilon \right)^{\frac{q-1}{q}} \left(\mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}} \left(\frac{1}{\varepsilon} \right)^{q-1} \right)^{\frac{1}{q}} \right] \\ &\leq k_1 C \left[\frac{q-1}{q} \varepsilon \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^q + \frac{1}{\varepsilon^{q-1} q} \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}} \right] \\ &= k_1 C \frac{q-1}{q} \varepsilon \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^q + k_1 C \frac{1}{\varepsilon^{q-1} q} \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}}. \end{aligned}$$

In the same manner for the integral $I_2(t)$ we can see that

$$\begin{aligned}
I_2(t) &= K \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^{q-p} \int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \\
&\leq K \left(\mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^{(q-p)\frac{q}{q-p}} \right)^{\frac{q-p}{q}} \left(\mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}} \right)^{\frac{p}{q}} \\
&\leq K \left(\mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^q \right)^{\frac{q-p}{q}} \left(\mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}} \right)^{\frac{p}{q}} \\
&= K \left(\mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^q \varepsilon \right)^{\frac{q-p}{q}} \left(\mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E N(ds, dz) \right)^q \left(\frac{1}{\varepsilon} \right)^{\frac{q-p}{p}} \right)^{\frac{p}{q}} \\
&\leq K \frac{q-p}{q} \varepsilon \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^q + K \frac{p}{q} \frac{1}{\varepsilon^{\frac{q-p}{q}}} \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E N(ds, dz) \right)^q
\end{aligned}$$

where we used Hölder's inequality in the first and fourth inequalities and Young's inequality in the third inequality.

It then follows that

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^q &\leq k_1 C \frac{q-1}{q} \varepsilon \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^q + k_1 C \frac{1}{\varepsilon^{q-1} q} \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}} \\
&\quad + K \frac{q-p}{q} \varepsilon \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^q + K \frac{p}{q} \frac{1}{\varepsilon^{\frac{q-p}{q}}} \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E N(ds, dz) \right)^q \\
&= \left(k_1 C \frac{q-1}{q} + K \frac{q-p}{q} \right) \varepsilon \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^q \\
&\quad + \left(k_1 C \frac{1}{\varepsilon^{q-1} q} + K \frac{p}{q} \frac{1}{\varepsilon^{\frac{q-p}{q}}} \right) \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}}.
\end{aligned}$$

Now we can choose a suitable number ε such that

$$\left(k_1 C \frac{q-1}{q} + K \frac{q-p}{q} \right) \varepsilon = \frac{1}{2}.$$

Consequently, there exists C which is independent of A such that

$$(4.8) \quad \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^q \leq C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}}.$$

Case II. Suppose $\xi \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$. Set $R(n, A) = (nI - A)^{-1}$, $n \in \mathbb{N}$. Then we put $\xi^n(t, \omega) = nR(n, A)\xi(t, \omega)$ on $[0, T] \times \Omega$. Since A is the infinitesimal generator of the C_0 -semigroup $S(t)$, $t \geq 0$ of contractions, by the Hille-Yosida Theorem, $\|R(n, A)\| \leq \frac{1}{n}$ and $\xi^n(t, \omega) \in \mathcal{D}(A)$, for every $(t, \omega) \in [0, T] \times \Omega$. Moreover, $\xi^n(t, \omega) \rightarrow \xi(t, \omega)$ pointwise on $[0, T] \times \Omega$. Also, we observe that $|\xi^n - \xi| = |nR(n, A)\xi - \xi| \leq 2|\xi|$. Therefore, it follows by applying the Lebesgue Dominated Convergence Theorem that

$$\int_0^T \int_Z |\xi^n(t, z) - \xi(t, z)|^p \nu(dz) dt$$

converges to 0 as $n \rightarrow \infty$, \mathbb{P} -a.s.. Since the poisson random measure N is a \mathbb{P} -a.s. positive measure and we have

$$\mathbb{E} \int_0^T \int_Z |\xi^n(t, z) - \xi(t, z)|^p N(dt, dz) = \mathbb{E} \int_0^T \int_Z |\xi^n(t, z) - \xi(t, z)|^p \nu(dz) dt,$$

we infer

$$\int_0^T \int_Z |\xi^n(t, z) - \xi(t, z)|^p N(dt, dz) \rightarrow 0, \quad \text{as } n \rightarrow \infty \text{ } \mathbb{P}\text{-a.s..}$$

One can also easily show that $\xi^n \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; \mathcal{D}(A))$.

Define, for each $n \in \mathbb{N}$, a process u^n by

$$u^n(t) = \int_0^t S(t-s) \xi^n(s, z) \tilde{N}(ds, dz), \quad t \in [0, T].$$

As we have already noted in case 1, function $u_n(t)$ can also be formulated in a way of strong solutions so that $u_n(t)$ is càdlàg for each $n \in \mathbb{N}$. By the discussion in case 1, for each $n \in \mathbb{N}$, $u^n(t)$, $0 \leq t \leq T$ satisfies the following

$$\mathbb{E} \sup_{0 \leq t \leq T} |u^n(t)|^q \leq C \mathbb{E} \left(\int_0^t \int_Z |\xi^n(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}}.$$

On the other hand, since by Theorem 2.7, we have

$$\begin{aligned} \mathbb{E} |u^n(t) - u(t)|_E^p &= \mathbb{E} |u^n(t) - u(t)|_E^p \\ &= \mathbb{E} \left| \int_0^t \int_Z \left(S(t-s) \xi^n(s, z) - S(t-s) \xi(s, z) \right) \tilde{N}(ds, dz) \right|_E^p \\ &\leq C_p \mathbb{E} \int_0^T \int_Z |\xi^n(s, z) - \xi(s, z)|^p \nu(dz) ds, \end{aligned}$$

we infer that $u^n(t)$ converges to $u(t)$ in $L^p(\Omega)$ for every $t \in [0, T]$. Moreover, from case 1, we know that

$$\mathbb{E} \sup_{0 \leq s \leq t} |u^n(s) - u^m(s)|_E^q \leq C \mathbb{E} \left(\int_0^t \int_Z |\xi^n(s, z) - \xi^m(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}}.$$

From above discussion, we know that the right hand-side of above inequality converges to 0 as $n, m \rightarrow \infty$. In this case, it is possible to construct a sequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_{n=1}^\infty$ for which the following is satisfied

$$\mathbb{E} \sup_{0 \leq s \leq T} |u^{n_{k+1}}(s) - u^{n_k}(s)|^q < \frac{1}{k^{2q+2}}.$$

Hence, on the basis of Chebyshev inequality, we obtain

$$\mathbb{P} \left\{ \sup_{0 \leq s \leq T} |u^{n_{k+1}}(s) - u^{n_k}(s)| > \frac{1}{k^2} \right\} \leq k^{2q} \mathbb{E} \sup_{0 \leq s \leq T} |u^{n_{k+1}}(s) - u^{n_k}(s)|^q < \frac{1}{k^2}.$$

Then the series $\sum_{k=1}^\infty \mathbb{P} \left\{ \sup_{0 \leq s \leq T} |u^{n_{k+1}}(s) - u^{n_k}(s)| > \frac{1}{k} \right\}$ will converges. It follows from the Borel-Cantelli Lemma that with probability 1 there exists an integer beyond which the inequality

$$\sup_{0 \leq s \leq T} |u^{n_{k+1}}(s) - u^{n_k}(s)| \leq \frac{1}{k^2}$$

holds. Consequently, the series of càdlàg functions

$$\sum_{k=1}^\infty [u^{n_{k+1}}(s) - u^{n_k}(s)]$$

converges uniformly on $[0, T]$ with probability 1 to a càdlàg function which we shall define by $\tilde{u} = (\tilde{u}(t))_{t \in [0, T]}$. Moreover, we have

$$\mathbb{E} \sup_{0 \leq t \leq T} |u^n(t) - \tilde{u}(t)|^q \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Therefore, by the Minkowski Inequality we have

$$\begin{aligned} \left[\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s)|^q \right]^{\frac{1}{q}} &\leq \left[\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s) - u^n(s)|^q \right]^{\frac{1}{q}} + \left[\mathbb{E} \sup_{0 \leq s \leq t} |u^n(s)|^q \right]^{\frac{1}{q}} \\ &\leq \left[\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s) - u^n(s)|^q \right]^{\frac{1}{q}} + \left[C \mathbb{E} \left(\int_0^t \int_Z |\xi^n(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}. \end{aligned}$$

Note that the constant C on the right hand side of above inequality does not depend on operator A . So the constant C remains the same for every n . It follows by letting $n \rightarrow \infty$ in above inequality that

$$\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s)|^q \leq C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}}$$

Also, we have for every $t \in [0, T]$, by Minkowski inequality that

$$\begin{aligned} (\mathbb{E} |\tilde{u}(t) - u(t)|_E^p)^{\frac{1}{p}} &\leq (\mathbb{E} |\tilde{u}(t) - u_n(t)|_E^p)^{\frac{1}{p}} + (\mathbb{E} |u(t) - u_n(t)|_E^p)^{\frac{1}{p}} \\ &\leq (\mathbb{E} |\tilde{u}(t) - u_n(t)|_E^q)^{\frac{1}{q}} + (\mathbb{E} |u(t) - u_n(t)|_E^p)^{\frac{1}{p}} \\ &\leq \left(\mathbb{E} \sup_{0 \leq t \leq T} |\tilde{u}(t) - u_n(t)|_E^q \right)^{\frac{1}{q}} + (\mathbb{E} |u(t) - u_n(t)|_E^p)^{\frac{1}{p}}. \end{aligned}$$

Letting $n \rightarrow \infty$, it follows that $u(t) = \tilde{u}(t)$ in $L^p(\Omega)$ for any $t \in [0, T]$. This shows the inequality (4.2) for $q' = q$. The case $q' > q$ follows from the fact that if the M -type p Banach space E satisfies Assumption 4.1 for some q , then Condition 1 is also satisfied with $q' > q$. □

The following result could be derived immediately from the proof of above theorem.

Corollary 4.1. *Let E be a M -type p Banach space, $1 < p \leq 2$ satisfying Assumption 4.1. Then the stochastic convolution process u has càdlàg modification.*

Corollary 4.2. *Let E be a M -type p Banach space, $1 < p \leq 2$. There exists a càdlàg modification \tilde{u} of u such that for some constant C and every stopping time $\tau > 0$ and $t > 0$,*

$$(4.9) \quad \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} |\tilde{u}(s)|_E^q \leq C \mathbb{E} \left(\int_0^{t \wedge \tau} \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q}{p}}.$$

Proof. Let us first consider the case when $\xi \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; \mathcal{D}(A))$. A similar argument as in Theorem 4.4 gives the following

$$\begin{aligned} \phi(u(t)) &= \int_0^t \phi'(u(s))(Au(s)) ds + \int_0^t \int_Z \phi'(u(s-))(\xi(s, z)) \tilde{N}(ds, dz) \\ &\quad + \int_0^t \int_Z \left[\phi(u(s-) + \xi(s, z)) - \phi(u(s-) - \phi'(u(s-))(\xi(s, z))) \right] N(ds, dz) \\ &\leq \int_0^t \int_Z \phi'(u(s-))(\xi(s, z)) \tilde{N}(ds, dz) \\ &\quad + \int_0^t \int_Z \left[\phi(u(s-) + \xi(s, z)) - \phi(u(s-) - \phi'(u(s-))(\xi(s, z))) \right] N(ds, dz) \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} \phi(u(s)) &= \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} |u(s \wedge \tau)|_E^q \\ &\leq \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} \int_0^{s \wedge \tau} \int_Z \phi'(u(r-))(\xi(r, z)) \tilde{N}(dr, dz) \\ &\quad + \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} \int_0^{s \wedge \tau} \left[\phi(u(r-) + \xi(r, z)) - \phi(u(r-) - \phi'(u(r-))(\xi(r, z))) \right] N(dr, dz) \\ &= \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} \int_0^s \int_Z 1_{(0, \tau]}(r) \phi'(u(r-))(\xi(r, z)) \tilde{N}(dr, dz) \\ &\quad + \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} \int_0^s \int_Z 1_{(0, \tau]}(r) \left| \phi(u(r-) + \xi(r, z)) - \phi(u(r-) - \phi'(u(r-))(\xi(r, z))) \right|_E N(ds, dz) \\ &= I_1 + I_2. \end{aligned}$$

Now we consider integral I_2 . By the definition of Lebesgue-Stieltjes integral, we have

$$\begin{aligned} &\int_0^s \int_Z \left| \phi(u(r-) + \xi(r, z)) - \phi(u(r-) - \phi'(u(r-))(\xi(r, z))) \right|_E 1_{(0, \tau]}(r) N(dr, dz) \\ &= \sum_{0 < r \leq s} \left| \phi(u(r-) + \xi(r, \xi(r))) - \phi(u(r-) - \phi'(u(r-))(\xi(r, \xi(r)))) \right|_E 1_{(0, \tau]}(r), \end{aligned}$$

Notice that the function $\phi(\cdot) = |\cdot|^q$ is of class C^2 . Applying Taylor formula to function ϕ we get for some $0 < \theta, \delta < 1$,

$$\begin{aligned} &\left| \phi(u(r-) + \xi(r, \pi(r))) - \phi(u(r-)) \right|_E 1_{(0, \tau]}(r) \\ &\leq |\xi(r, \pi(r))|_E \left| \phi'(u(r-) + \theta \xi(r, \pi(r))) \right|_E 1_{(0, \tau]}(r), \\ &\left| \phi(u(r-) + \xi(r, \pi(r))) - \phi(u(r-) - \phi'(u(r-))(\xi(r, \pi(r)))) \right|_E 1_{(0, \tau]}(r) \\ &\leq \frac{1}{2} |\xi(r, \pi(r))|_E^2 |\phi''(u(r-)) + \delta \xi(r, \pi(r))|_E 1_{(0, \tau]}(r) \end{aligned}$$

Moreover we know that $|\phi'(x)|_{\mathcal{L}(E)} \leq k_1 |x|_E^{q-1}$, so we obtain

$$\begin{aligned} \left| \phi'(u(r-) + \theta \xi(r, \pi(r))) \right|_E 1_{(0, \tau]}(r) &\leq k_1 |u(r-) + \theta \xi(r, \pi(r))|_E^{q-1} 1_{(0, \tau]}(r) \\ &\leq k_1 \max \{ |u(r-)|_E^{q-1} 1_{(0, \tau]}(r), |u(r-) + \xi(r, \pi(r))|_E^{q-1} 1_{(0, \tau]}(r) \}. \end{aligned}$$

Observe that

$$|u(r-)|_E^{q-1} 1_{(0, \tau]}(r) \leq \sup_{0 \leq r \leq s} |u(r-)|_E^{q-1} 1_{(0, \tau]}(r) \leq \sup_{0 \leq s \leq t \wedge \tau} |u(s-)|_E^{q-1} = \sup_{0 \leq s \leq t \wedge \tau} |u(s)|_E^{q-1},$$

and

$$|u(r-) + \xi(r, \pi(r))|_E^{q-1} 1_{(0, \tau]}(r) \leq \sup_{0 \leq r \leq s} |u(r)|_E^{q-1} 1_{(0, \tau]}(r) \leq \sup_{0 \leq s \leq t \wedge \tau} |u(s)|_E^{q-1},$$

where $q \geq 2$. Therefore, we infer

$$\begin{aligned} \left| \phi(u(r-) + \xi(r, \pi(r))) - \phi(u(r-)) \right|_E 1_{(0, \tau]}(r) &\leq |\xi(r, \pi(r))|_E 1_{(0, \tau]}(r) \left| \phi'(u(r-) + \theta \xi(r, \pi(r))) \right|_{\mathcal{L}(E)} \\ &\leq k_1 |\xi(r, \pi(r))|_E 1_{(0, \tau]}(r) \sup_{0 \leq s \leq t \wedge \tau} |u(s)|_E^{q-1}. \end{aligned}$$

Similarly, from the assumption $|\phi''(x)| \leq k_2 |x|_E^{q-2}$ we obtain

$$|\phi''(u(r-)) + \delta \xi(r, \pi(r))|_E 1_{(0, \tau]}(r) \leq k_2 \sup_{0 \leq s \leq t \wedge \tau} |u(s)|_E^{q-2} 1_{(0, \tau]}(r).$$

It then follows that

$$\begin{aligned} \sum_{0 < r \leq s} \left| \phi(u(r-) + \xi(r, \xi(r))) - \phi(u(r-)) - \phi'(u(r-))(\xi(r, \xi(r))) \right|_E 1_{(0, \tau]}(r) \\ = \sum_{0 < r \leq s} \left| \phi(u(r-) + \xi(r, \xi(r))) - \phi(u(r-)) - \phi'(u(r-))(\xi(r, \xi(r))) \right|_E^{(2-p)+(p-1)} 1_{(0, \tau]}(r) \\ \leq \left(2k_1 |\xi(r, \pi(r))|_E \sup_{0 \leq s \leq t \wedge \tau} |u(s)|_E^{q-1} 1_{(0, \tau]}(r) \right)^{2-p} \left(k_2 |\xi(r, \pi(r))|_E^2 \sup_{0 \leq s \leq t \wedge \tau} |u(s)|_E^{q-2} 1_{(0, \tau]}(r) \right)^{p-1} \\ = K \sup_{0 \leq s \leq t \wedge \tau} |u(s)|_E^{q-p} \sum_{0 < r \leq s} |\xi(r, \pi(r))|_E^p 1_{(0, \tau]}(r). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^s \int_Z \left| \phi(u(r-) + \xi(r, z)) - \phi(u(r-)) - \phi'(u(r-))(\xi(r, z)) \right|_E 1_{(0, \tau]}(r) N(dr, dz) \\ \leq K \sup_{0 \leq s \leq t \wedge \tau} |u(s)|_E^{q-p} \int_0^s \int_Z |\xi(r, z)|_E^p 1_{(0, \tau]}(r) N(dr, dz). \end{aligned}$$

Hence, for integral I_2 , we can estimate as follows

$$I_2 \leq K \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} |u(s)|_E^{q-p} \int_0^s \int_Z |\xi(r, z)|_E^p 1_{(0, \tau]}(r) N(dr, dz).$$

For integral I_1 , applying the stopped Davis' inequality yields the following

$$\begin{aligned} I_1 &\leq C \mathbb{E} \left(\int_0^s \int_Z |\phi'(u(r-))(\xi(r, z))|_E^p 1_{(0, \tau]}(r) N(dr, dz) \right)^{\frac{1}{p}} \\ &\leq k_1 C \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} |u(s)|_E^{q-1} \left(\int_0^{t \wedge \tau} \int_Z |\xi(r, z)|_E^p N(ds, dz) \right)^{\frac{1}{p}}. \end{aligned}$$

The rest argument goes without any difference with the proof of Theorem 4.4. □

Theorem 4.5. *Let E be an M -type p Banach space, $1 < p \leq 2$, satisfying Assumption 4.1. Then there exists a constant C such that for every process u there exists a càdlàg modification \tilde{u} of u such that for all $0 \leq t \leq T$ and $0 < q' < \infty$,*

$$(4.10) \quad \mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s)|_E^{q'} \leq C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q'}{p}},$$

Proof. The inequality (4.10) has already been shown for $q' \geq q$ in Theorem 4.4. Now we are in a position to show it for $0 < q' < p$. Let us fix q' such that $0 < q' < q$. Take $\lambda > 0$. Define a stopping time

$$\tau := \inf \left\{ t : \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{1}{p}} > \lambda \right\}.$$

Since the process $\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz)$, $0 < t \leq T$ is right continuous, the random time τ is indeed a \mathcal{F}_{t+} -stopping time. Moreover, we find that $\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \leq \lambda$, for $0 < t < \tau$, and $\int_0^\tau \int_Z |\xi(s, z)|_E^p N(ds, dz) \geq \lambda$ when $\tau < \infty$. Also, we observe that for every $0 < t \leq T$,

$$(4.11) \quad \mathbb{E} \int_0^t \int_Z f(s, z) \tilde{N}(ds, dz) = \mathbb{E} \int_0^{t-} \int_Z f(s, z) \tilde{N}(ds, dz).$$

This equality can be verified first for step functions, then for every function f in $\xi \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$ we can approximate it by step functions in $\mathcal{M}_{step}^p([0, T] \times \Omega \times Z; E)$, so the equality (4.11) holds for every $f \in \xi \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$. Therefore, by using Chebyshev's inequality and Corollary 4.2 to Theorem 4.4, we obtain

$$(4.12) \quad \begin{aligned} \mathbb{P} \left(\sup_{0 \leq s \leq t \wedge \tau} |u(s)| > \lambda \right) &\leq \frac{1}{\lambda^q} \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} |u(s)|^q \\ &\leq \frac{C}{\lambda^q} \mathbb{E} \left(\int_0^{t \wedge \tau} \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{q}{p}} \\ &= \frac{C}{\lambda^q} \mathbb{E} \left(\int_0^{(t \wedge \tau)-} \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{q}{p}} \\ &\leq \frac{C}{\lambda^q} \mathbb{E} \left[\left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{q}{p}} \wedge \lambda^q \right]. \end{aligned}$$

On the other hand, since $\{\sup_{0 \leq s \leq t} |u(s)| > \lambda, \tau \geq t\} \subset \{\sup_{0 \leq s \leq t \wedge \tau} |u(s)| > \lambda\}$, we have

$$(4.13) \quad \begin{aligned} \mathbb{P} \left(\sup_{0 \leq s \leq t} |u(s)| > \lambda \right) &= \mathbb{P} \left(\sup_{0 \leq s \leq t} |u(s)| > \lambda, \tau \geq t \right) + \mathbb{P} \left(\sup_{0 \leq s \leq t} |u(s)| > \lambda, \tau < t \right) \\ &\leq \mathbb{P} \left(\sup_{0 \leq s \leq t} |u(s)| > \lambda, \tau \geq t \right) + \mathbb{P}(\tau < t) \\ &\leq \mathbb{P} \left(\sup_{0 \leq s \leq t \wedge \tau} |u(s)| > \lambda \right) + \mathbb{P}(\tau < t). \end{aligned}$$

Substituting (4.12) into (4.13) results in

$$\begin{aligned} \mathbb{P} \left(\sup_{0 \leq s \leq t} |u(s)| > \lambda \right) &\leq \frac{C}{\lambda^q} \mathbb{E} \left[\left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{q}{p}} \wedge \lambda^q \right] \\ &\quad + \mathbb{P} \left[\left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{1}{p}} > \lambda \right]. \end{aligned}$$

Integrating both sides of the last inequality with respect to measure $q'\lambda^{q'-1}d\lambda$ and applying the equality $\mathbb{E}|X|^{q'} = \int_0^\infty q'\lambda^{q'-1}\mathbb{P}(|X| > \lambda)d\lambda$, see [8], we infer that

$$\begin{aligned}
(4.14) \quad \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|^{q'} &= \int_0^\infty \mathbb{P}(\sup_{0 \leq s \leq t} |u(s)| > \lambda) q' \lambda^{q'-1} d\lambda \\
&\leq \int_0^\infty \frac{C}{\lambda^q} \mathbb{E} \left[\left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{q}{p}} \wedge \lambda^q \right] q' \lambda^{q'-1} d\lambda \\
&\quad + \int_0^\infty \mathbb{P} \left[\left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{1}{p}} > \lambda \right] q' \lambda^{q'-1} d\lambda \\
&= \int_0^\infty \frac{C}{\lambda^q} \mathbb{E} \left[\left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{q}{p}} \wedge \lambda^q \right] q' \lambda^{q'-1} d\lambda \\
&\quad + \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{q'}{p}}.
\end{aligned}$$

Let us denote $\left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{1}{p}}$ by X . The first term on the right hand side of (4.14) becomes

$$\begin{aligned}
&\frac{C}{\lambda^q} \int_0^\infty \mathbb{E} \left[\left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{q}{p}} \wedge \lambda^q \right] q' \lambda^{q'-1} d\lambda \\
&= C \int_0^\infty \mathbb{E}(X^q \wedge \lambda^q) q' \lambda^{q'-q-1} d\lambda \\
&= C \mathbb{E} \int_0^\infty (X^q \wedge \lambda^q) q' \lambda^{q'-q-1} d\lambda \\
&= C \mathbb{E} \int_0^X \lambda^q q' \lambda^{q'-q-1} d\lambda + C \mathbb{E} \int_X^\infty |X|^q q' \lambda^{q'-q-1} d\lambda \\
&= C \mathbb{E} X^{q'} + C \mathbb{E} X^q \int_X^\infty q' \lambda^{q'-q-1} d\lambda \\
&= C \left(1 + \frac{q'}{q - q'} \right) \mathbb{E} X^{q'} \\
&= \frac{Cq}{q - q'} \mathbb{E} X^{q'} \\
&= \frac{Cq}{q - q'} \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{q'}{p}}
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} |u(s)|^{q'} &\leq \frac{Cq}{q - q'} \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{q'}{p}} + \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{q'}{p}} \\
&= \left(1 + \frac{Cq}{q - q'} \right) \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|^p N(ds, dz) \right)^{\frac{q'}{p}},
\end{aligned}$$

which completes the proof. \square

Corollary 4.3. *Let E be an M -type p Banach space, $1 < p \leq 2$ satisfying Assumption 4.1. Then there exists an E -valued càdlàg modification \tilde{u} of u such that for some constant $C > 0$, independent*

of u , and all $t \in [0, T]$ and $0 < q' \leq p$,

$$(4.15) \quad \mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s)|_E^{q'} \leq C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p \nu(dz) ds \right)^{\frac{q'}{p}}$$

Proof of Corollary 4.3. First, we consider the case $q' = p$. Since $\xi \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$, so both integrals $\int_0^t \int_Z |\xi(s, z)|_E^p \nu(dz) ds$ and $\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz)$ are well defined as Lebesgue-Stieltjes integrals. We can obtain from Theorem 4.5 with $q' = p$ that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |u(s)|_E^p &\leq C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right) \\ &= C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p \nu(dz) ds \right). \end{aligned}$$

This shows (4.15) for $q' = p$. Now we are in a position to show Inequality (4.15) for $0 < q' < p$. Let q' be fixed. Take $\lambda > 0$. Define stopping time

$$\tau = \inf \{ t \in [0, T] : \left(\int_0^t \int_Z |\xi(s, z)|^p \nu(dz) ds \right)^{\frac{1}{p}} > \lambda \}.$$

The random variable τ is a stopping time. Indeed the process $\int_0^t \int_Z |\xi(s, z)|^p \nu(dz) ds$, $0 \leq t \leq T$ is a continuous process and so the claim follows immediately. It follows from Chebyshev's inequality and Corollary 4.2 that

$$\begin{aligned} (4.16) \quad \mathbb{P} \left(\sup_{0 \leq s \leq t \wedge \tau} |u(s)| > \lambda \right) &= \mathbb{E} 1_{\{\sup_{0 \leq s \leq t \wedge \tau} |u(s)| > \lambda\}} \\ &\leq \frac{1}{\lambda^q} \mathbb{E} \sup_{0 \leq s \leq t \wedge \tau} |u(s)|^q \\ &\leq \frac{C}{\lambda^q} \mathbb{E} \left(\int_0^{t \wedge \tau} \int_Z |\xi(s, z)|^p \nu(dz) ds \right)^{\frac{q}{p}} \\ &\leq \frac{C}{\lambda^q} \mathbb{E} \left[\left(\int_0^t \int_Z |\xi(s, z)|^p \nu(dz) ds \right)^{\frac{q}{p}} \wedge \lambda^q \right], \end{aligned}$$

where we used the definition of stopping time τ and the increasing property of process $\int_0^t \int_Z |\xi(s, z)|^p \nu(dz) ds$, $0 \leq t \leq T$. The rest of the proof can be done exactly in the same manner as in the proof of Theorem 4.5. \square

Corollary 4.4. *Let E be an M -type p Banach space, $1 < p \leq 2$ satisfying Assumption 4.1. Then for any $n \in \mathbb{N}$ there exists a constant $C = C(n)$ such that for every every $\xi \in \bigcap_{k=1}^n \mathcal{M}^{p^k}([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$ and $t \in [0, T]$ we have*

$$(4.17) \quad \mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s)|_E^{p^n} \leq C \sum_{k=1}^n \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^{p^k} \nu(dz) ds \right)^{p^{n-k}}.$$

where \tilde{u} is the càdlàg modification of u as before.

The proof of Corollary 4.4 is similar to the proof Lemma 5.2 in Bass and Cranston [2] or of Lemma 4.1 in Protter and Talay [16]. Essential ingredients of that proof are the following two results. The first of them being about integration of real valued processes.

Lemma 4.6. *Let E be an M -type p Banach space, $1 < p \leq 2$, satisfying Assumption 4.1. For any $0 < q' < \infty$, there exists a constant C such that for all $\xi \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$ we have*

$$(4.18) \quad \mathbb{E} \sup_{0 \leq s \leq t} \left(\int_0^s \xi(r, z) \tilde{N}(dr, dz) \right)_E^{q'} \leq C \mathbb{E} \left(\int_0^t |\xi(s, z)|_E^p N(ds, dz) \right)^{\frac{q'}{p}}, \quad t \in [0, T].$$

Proof of Lemma 4.6. This result is a special case of Theorem 4.5 when $S(t) = I$, $0 \leq t \leq T$. \square

Lemma 4.7. *For any $n \in \mathbb{N}$ there exists a constant $D_n > 0$ such for any process*

$$f \in \bigcap_{k=1}^n \mathcal{M}^{p^k}([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; \mathbb{R})$$

and $t \in [0, T]$, the following inequality

$$(4.19) \quad \mathbb{E} \sup_{0 \leq s \leq t} \left(\int_0^s \int_Z f(r, z) \tilde{N}(dr, dz) \right)^{p^n} \leq D_n \sum_{k=1}^n \mathbb{E} \left(\int_0^t \int_Z |f(s, z)|^{p^k} \nu(dz) ds \right)^{p^{n-k}}$$

holds.

Proof of Lemma 4.7. We shall show this Lemma by induction. The case $n = 1$. This follows from [3]. Now we assume that the assertion in the Claim is true for $n - 1$, where $n \in \mathbb{N}$ and $n \geq 2$. We will show that it is true for n . Since by assumption $f \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; \mathbb{R})$, so both integrals $\int_0^t \int_Z |f(s, z)|^p N(ds, dz)$ and $\int_0^t \int_Z |f(s, z)|^p \nu(dz) ds$ are well defined as Lebesgue-Stieltjes integrals. Moreover, we have

$$(4.20) \quad \int_0^t \int_Z |f(s, z)|^p \tilde{N}(ds, dz) = \int_0^t \int_Z |f(s, z)|^p N(ds, dz) - \int_0^t \int_Z |f(s, z)|^p \nu(dz) ds.$$

Hence by applying first inequality (4.18) and next the equality (4.20) we infer that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s \int_Z f(r, z) \tilde{N}(dr, dz) \right|^{p^n} &\leq C \mathbb{E} \left| \int_0^t \int_Z |f(s, z)|^p N(ds, dz) \right|^{p^{n-1}} \\ &\leq 2^{p^{n-1}} C \left\{ \mathbb{E} \left(\int_0^t \int_Z |f(s, z)|^p \tilde{N}(ds, dz) \right)^{p^{n-1}} + \mathbb{E} \left(\int_0^t \int_Z |f(s, z)|^p \nu(dz) ds \right)^{p^{n-1}} \right\}. \end{aligned}$$

Next, by the inductive assumption applied to the real valued process $|f|^p \in \bigcap_{k=1}^{n-1} \mathcal{M}^{p^k}([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; \mathbb{R})$, we have

$$\begin{aligned} \mathbb{E} \left| \int_0^t \int_Z f(s, z) \tilde{N}(ds, dz) \right|^{p^n} &\leq 2^{p^{n-1}} C \left(D_{n-1} \sum_{i=1}^{n-1} \mathbb{E} \left(\int_0^t \int_Z |f(s, z)|^{p^{i+1}} \nu(dz) ds \right)^{p^{n-1-i}} + \mathbb{E} \left(\int_0^t \int_Z |f(s, z)|^p \nu(dz) ds \right)^{p^{n-1}} \right) \\ &\leq D_n \sum_{k=1}^n \mathbb{E} \left(\int_0^t \int_Z |f(s, z)|^{p^k} \nu(dz) ds \right)^{p^{n-k}}. \end{aligned}$$

This proves the validity of the assertion in the Lemma for n what completes the whole proof. \square

Proof of Corollary 4.4. Let us take $n \in \mathbb{N}$. By applying first Theorem 4.5 and next the equality (4.20) when $\xi \in \mathcal{M}^p([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; E)$, we infer that for all $t \in [0, T]$,

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} |\tilde{u}(s)|_E^{p^n} &\leq C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p N(ds, dz) \right)^{p^{n-1}} \\
&\leq 2^{p^{n-1}} C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p \tilde{N}(ds, dz) \right)^{p^{n-1}} \\
&\quad + 2^{p^{n-1}} C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p \nu(dz) ds \right)^{p^{n-1}} \\
&\leq 2^{p^{n-1}} C D_{n-1} \sum_{k=1}^{n-1} \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^{p^{k+1}} \nu(dz) ds \right)^{p^{n-1-k}} \\
&\quad + 2^{p^{n-1}} C \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^p \nu(dz) ds \right)^{p^{n-1}} \\
&\leq C(n) \sum_{k=1}^n \mathbb{E} \left(\int_0^t \int_Z |\xi(s, z)|_E^{p^k} \nu(dz) ds \right)^{p^{n-k}},
\end{aligned}$$

where we used in the third inequality Lemma 4.6 with f replaced by real-valued process $|\xi|_E^p \in \bigcap_{k=1}^{n-1} \mathcal{M}^{p^k}([0, T] \times \Omega \times Z, \hat{\mathcal{P}}, \lambda \times \mathbb{P} \times \nu; \mathbb{R})$. This completes the proof of Corollary 4.4. \square

Acknowledgements 1. *Preliminary versions of this work were presented at the First CIRM-HCM Joint Meeting on Stochastic Analysis and SPDE's which was held at Trento (January 2010). The research of the third named author was partially supported by an ORS award at the University of York. Results presented in this article will be included in the Ph. D. thesis the third named author. This work was supported by the FWF-Project P17273-N12. Part of the work was done at the Newton institute for Mathematical Sciences in Cambridge (UK), whose support is gratefully acknowledged, during the program "Stochastic Partial Differential Equations".*

REFERENCES

- [1] D. Applebaum, *Lévy processes and stochastic calculus*, Cambridge Studies in Advanced Mathematics, **93**, Cambridge University Press, Cambridge, (2004)
- [2] R. F. Bass and M. Cranston, *The Malliavin calculus for pure jump processes and applications to local time*, Ann. Probab. **14**, no. 2, 490–532 (1986).
- [3] Z. Brzeźniak, E. Hausenblas, *Maximal regularity for stochastic convolutions driven by Lévy processes*, Probab. Theory Related Fields **145**, no. 3-4, 615–637 (2009)
- [4] Z. Brzeźniak, S. Peszat, *Maximal inequalities and exponential estimates for stochastic convolutions in Banach spaces*, Stochastic processes, physics and geometry: new interplays, I (Leipzig, 1999), 55–64, CMS Conf. Proc., **28**, Amer. Math. Soc., Providence, RI, (2000)
- [5] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Encyclopedia of Mathematics and its Applications, **44**, Cambridge University Press, Cambridge, (1992)
- [6] Paul R., Halmos, *Measure Theory*, D. Van Nostrand Company, Inc., New York, N. Y., (1950)
- [7] E. Hausenblas, *A note on the Itô formula of stochastic integrals in Banach spaces*, Random Oper. Stochastic Equations **14**, no. 1, 45–58, (2006)
- [8] A. Ichikawa, *Some inequalities for martingales and stochastic convolutions*, Stochastic Anal. Appl. **4**, no. 3, 329–339, (1986)
- [9] A.N. Kolmogorov, S.V. Fomin, *Elementi di teoria delle funzioni e di analisi funzionale*, Edizione MIR, Moscau, Translation from the russian: *Elementy teorii funktsij i funktsional'nogo*, Copyright by NAUKA, Moscau, (1980)
- [10] P. Kotelenez, *A submartingale type inequality with applications to stochastic evolution equations*, Stochastics **8**, no. 2, 139–151, (1982/83)

- [11] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, **44**, Springer-Verlag, New York, (1983)
- [12] S. Peszat and J. Zabczyk. *Stochastic partial differential equations with Levy noise*. Encyclopedia of Mathematics and its Applications, 113. Cambridge University Press, Cambridge, 2007.
- [13] Tubaro, L. An estimate of Burkholder type for stochastic processes defined by the stochastic integral. *Stochastic Anal. Appl.* **2**, no. 2, 187–192 (1984)
- [14] B. Rüdiger, *Stochastic integration with respect to compensated Poisson random measures on separable Banach spaces*, *Stoch. Stoch. Rep.* **76**, no. 3, 213–242 (2004)
- [15] B. Rüdiger, G. Ziglio, *Itô formula for stochastic integrals w.r.t. compensated Poisson random measures on separable Banach spaces*, *Stochastics* **78**, no. 6, 377–410 (2006)
- [16] P. Protter and D. Talay, *The Euler scheme for Lévy driven stochastic differential equations*, *Ann. Probab.* **25**, 1, 393–423 (1997)
- [17] A. D. Wentzell, *A course in the theory of stochastic processes*, Translated from the Russian by S. Chomet. With a foreword by K. L. Chung, McGraw-Hill International Book Co., New York, (1981).

* ZDZISŁAW BRZEŹNIAK, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF YORK, HESLINGTON, YORK YO10 5DD, UK, E-MAIL: ZB500@YORK.AC.UK

* ERIKA HAUSENBLAS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SALZBURG, HELLBRUNNERSTR. 34, 5020 SALZBURG, AUSTRIA, E-MAIL: ERIKA.HAUSENBLAS@SBG.AC.AT

† JIAHUI ZHU, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF YORK, HESLINGTON, YORK, YO10 5DD, UK, E-MAIL: JZ527@YORK.AC.UK